# PROJECTIVE GEOMETRY WITH COORDINATES FROM A COMMUTATIVE PRIMARY RING

BY

ROBERT WILDER GIBSON

A.B., Fort Hays Kansas State College, 1938
A.M., University of Illinois, 1939

AN ABSTRACT OF A THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
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MATHEMATICS IN THE GRADUATE SCHOOL OF
THE UNIVERSITY OF ILLINOIS, 1943

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It has been shown by R. Baer, "A Unified Theory of Projective Spaces and Finite Abelian Groups," Transactions of the American Mathematical Society, 52:2, Sept., 1942, that there exists a projective geometry over a primary ring, exactly as over a field. It is the purpose of this investigation to develope the concept cross-ratio for this projective geometry. In classical geometry it has been shown that cross-ratio theory is unsatisfactory if the coordinates are taken from a non-commutative field. So we consider a commutative primary ring.

1. **Preliminary.** The elements of the geometry are the admissible subgroups (linear subspaces $A=A+A=AR$) of an abelian group G with multiplicative operators from a commutative primary ring R. The associative and distributive laws hold for R and for operation on G. R has an identity 1, which is an identity operator. The ideals of R form a finite monotone (proper) chain; let $R=P^0$ and $P^m=0$, where $m=1$ for the classical geometry with R a field.

There is a number p in R such that the ideals $P^n=p^nR$; we select such a p, called a prime, for future computations. For any r in R, there is a representation $r=r''p^{r'}$, where $r'$ is unique; we adopt this notation. r has a (unique) inverse if and only if $r'=0$.

The points of the geometry are the cyclic subgroups gR, and g is called a generator of gR. The order of an element g or of a point gR is the smallest n for which $gP^n=0$.

If the group A is generated by a finite number of elements, there is an independent set of non-zero elements which generate A, a basis of A; and their orders form a set of invariants which characterize the group. The order of A is the sum n(A) of the invariants, i.e., the sum of the orders of any independent set of generators. We admit zeros (of G) to an independent set. The dimension k(A) is one less than the number of elements in a basis.

A line is a group of dimension 1. A plane has dimension 2. The null element has dimension $-1$, and order 0. We shall consider only groups with a finite number of generators.

2. **Representations.** If the elements $a_j$ are in the group $\Sigma b_iR$, then there are numbers $m_{ij}$ in R such that $a_j=\Sigma b_im_{ij}$, or in matrix notation $a=bM$. If M is a square matrix, $|M|$ is its determinant. The matrix M is not uniquely determined by a and b; one of our main problems is to derive invariants for the sets of elements a and b from the matrices M.

If $a=bM$ and $b=cN$, then $a=c(NM)$.

If $\Sigma_{i=0}^{k} a_i R$ is of dimension k and if $b=aM$ for M a square matrix, then a necessary and sufficient condition for $\Sigma a_i R=\Sigma b_i R$ is that $|M|'=0$.

For any set of elements a there is a matrix M of determinant 1 such that aM is an independent set.

If the element c is in $\Sigma_{i=0}^{k} b_i R$ of dimension k, if $b=aM$ where M is a square matrix, and if $c=aN$, then there are $q_i$ with matrix Q such that $c=bQ$ and $N=MQ$.

If $\Sigma c_i R \leqq \Sigma b_i R \leqq \Sigma a_i R$, then the set of determinants $|M|$ such that $c=aM$ is equal to the set of products $|H|\cdot|K|$ for $c=bH$ and $b=aK$.

3. **Deficiency and modulus.** If $\Sigma_{i=0}^{h} b_i R \leqq \Sigma_{i=0}^{h} a_i R$, then the deficiency (b/a) is the minimum of the exponents $|M|'$ for $b=aM$; the minimum of the exponents $(|M|-|N|)'$ for $b=aM=aN$ is called $e=e(b:a)$ the exponent of the modulus.

The set of values of $|M|$ for $b=aM$ is a coset of R modulo $P^e$ where $e=e(b:a)$.

$(b/a)=\min\ [m,\ n(\Sigma aR)-n(\Sigma bR)]$ and $e(b:a)=\min\ [m,\ n(\Sigma aR)-n(\Sigma bR)+v]$, where v is the least invariant of $\Sigma_{i=0}^{h} b_i R$ if it is of dimension h and where v is zero otherwise.

4. **Ratio.** The presence of zero divisors (i.e., P) necessitates the definition of a ratio as an ordered number pair. The product of two ratios is $(x,y)(r,s)=(xr,ys)$. Also $(x,y)r=(xr,yr)$. Ratios are equal if each is a multiple of the other (by a number in R). We define $1-(x,y)=(y-x,y)$.

The set of numbers $1+P^q$ forms a multiplicative group when q is greater than zero. If r and s are numbers in R, then we call $r(1+P^q)$ and $(r[1+P^q],s)$ rays; q is called an index of the ray; q is unique if $q+r'$ and $q+s'$ are less than m. A ray is proper if q is greater than zero or if r is zero.

The product of two rays is a ray; the minimum of the indices is an index.

The set of ratios $(u(x_1, \ldots, x_k), v(x_1, \ldots, x_k))$, where u and v are linear in each $x_i$, $x_i$ in R, is a ray if $v(x)-v(0)$ is in $v(0)P$. The ray is proper if the same holds also for u; and it has index greater than or equal to the minimum of $[v(x)-v(0)]'-v(0)'$ and $[u(x)-u(0)]'-u(0)'$.

5. **Cross-ratio.** If $eR+fR$ has dimension 1, if a, b, c, d are in it, and if $(abcd)=(ef)M$, then we denote by $(x,y/e,f:M)$ the submatrix

of M satisfying $(xy)=(ef)(x,y/e,f:M)$ where x, y are any two of the elements a, b, c, d. The cross-ratio is the set of ratios $(a,b:c,d/e,f)$ $=(|(a,c/e,f:M)|\cdot|(b,d/e,f:M)|, |(a,d/e,f:M)|\cdot|(b,c/e,f:M)|)$ for all M such that $(abcd)=(ef)M$.

$(a,b:c,d/e,f) = (a^*,b^*:c^*,d^*/e^*,f^*)$ if $aR=a^*R$, ..., $dR=d^*R$, and $eR+fR=e^*R+f^*R$; that is, the value of a cross-ratio depends only upon the four points and the line in which they lie.

For four points on a line, there are the customary relations among the twenty-four cross-ratios.

If the line H is contained in the line L and if points A, B, C, D are in H, then $(A,B:C,D/L) = (A,B:C,D/H)p^{2[n(L)-n(H)]}$.

If a and d, and b and c generate lines, then the cross-ratio (a,b:c,d) is a ray. If a and c, and b and d also generate lines, the ray is proper; otherwise there is a value with the first member zero.

An index of a cross-ratio which is a proper ray is (can be taken) greater than or equal to the minimum of the lesser invariants of the lines generated by a point from each pair.

If a cross-ratio is a proper ray containing (r,s), then it is a one-element ray if and only if m is less than or equal to the index of the ray plus the maximum of r′ or s′.

6. **Menelaus.** Three or more points are collinear if any two of them generate the same line. The theorem of Menelaus can be extended in our projective geomtry to: Given a triangle with three independent vertices $V_1$, $V_2$, $V_3$ and two triples of collinear points $A_i$ and $B_i$, such that $A_i$, $B_i$, $V_j$, $V_k$ are collinear for $j\neq i\neq k$, then $(V_2,V_3:A_1,B_1)(V_3,V_1:A_2,B_2)(V_1,V_2:A_3,B_3)$ is a proper ray containing the ratio (1,1).

7. **Projectivities.** A projectivity (one-to-one incidence preserving map of subgroups between two groups) can be induced by a linear transformation of the elements if and only if it preserves cross-ratio.

A (proper) semi-linear transformation of a group with two independent elements of order m is linear if and only if it preserves cross-ratio.

These results should be compared, with each other and with the previous result of Baer: A projectivity can be induced by a semi-linear transformation if the group has three independent elements of maximum order. In this theorem the hypothesis of the existence of three independent elements of maximum order is indispensible, whereas no such hypothesis is needed when cross-ratio is preserved.

## VITA

Robert Wilder Gibson was born on December 19, 1917 in Greensburg, Kansas. He received his elementary and secondary education in the public schools of Greensburg, and was graduated in 1935 as valedictorian of his high school class. He attended Fort Hays Kansas State College from 1935 to 1938, and was valedictorian of the class of 1938, when he received the degree Bachelor of Arts. He was one of the five co-winners of the William Lowell Putnam Mathematical Competition in 1938. He was a graduate student at the University of Illinois from 1938 to 1943, and he received the degree Master of Arts from this university in 1939. In 1938 and 1939 he received a graduate scholarship and a graduate fellowship from the University of Illinois, and from 1940 to 1943 he was an assistant in mathematics at this institution. He has also taught a course and substituted for instructors in physics and mathematics at Fort Hays Kansas State College. He is a member of the National Honor Society, F.H.K.S.C. Honor Society, Delta Epsilon, Sigma Pi Sigma, Pi Mu Epsilon, Sigma Xi, American Mathematical Society, Institute of Mathematical Statistics, and the American Association for the Advancement of Science; and he has served as treasurer of the Cosmopolitan Club of the University of Illinois and as national treasurer of The American College Quill Club.

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